

NNLO corrections for improving the accuracy of light quark mass determinations

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- I. Introduction & Motivation
- II. Concepts & Framework
- III. Results & Discussion
- IV. Summary & Conclusion

Based on

arXiv:1004.4613 [hep-ph] , Phys. Rev. D 80, 014501 (2009)
in collaboration with: L. G. Almeida, Y. Aoki, N.H. Christ,
T. Izubuchi, C.T. Sachrajda and A. Soni

Introduction

- Light quark masses (up-, down- and strange-quark masses) can be determined non-perturbatively with lattice simulations in QCD
- Result from RBC/UKQCD Coll., domain-wall fermions:

$$m_{ud}^{\overline{\text{MS}}}(2 \text{ GeV}) = 3.72(0.16)_{\text{stat}}(0.18)_{\text{syst}}(0.33)_{\text{ren}} \text{ MeV}$$

$$m_s^{\overline{\text{MS}}}(2 \text{ GeV}) = 107.3(4.4)_{\text{stat}}(4.9)_{\text{syst}}(9.7)_{\text{ren}} \text{ MeV}$$

C. Allton et al.

- Error (11%) from renormalization dominates (>60% of tot.)
- Pert. calculations are performed in dim. reg.
~~ not directly amenable to lattice calculations
- Direct calculation of bare quantity with lattice spacing acting as ultra-violet cutoff in some particular discretization of QCD instead of space-time dimension $d \neq 4$
- Minimal subtraction(MS) à la dim. reg. not directly possible

Introduction

Regularization invariant momentum subtraction schemes

- Use for renormalization regularization invariant(RI) scheme, which removes ultraviolet divergences at a certain momentum point(subtraction point) \rightsquigarrow RI/MOM-scheme
Martinelli et al. '93-'95
- Determine QCD parameters: $m_R = Z_m m_B$, $\Psi_R = Z_q^{1/2} \Psi_B$, ..
 \rightsquigarrow fix renormalization constants, define scheme in PT:

$$S_R^{-1} = Z_q^{-1} S_B^{-1} \propto p \Sigma_R^V(p^2) - m_R \Sigma_R^S(p^2) \quad \Leftrightarrow \quad \text{Diagram: } \frac{1}{p} + \frac{1}{p} \text{ (with loop)} + \dots$$

$$\lim_{m_R \rightarrow 0} \frac{1}{12m_R} \text{Tr}[S_R^{-1}(p)] \Big|_{p^2 = -\mu^2} = 1 \quad \lim_{m_R \rightarrow 0} \frac{1}{48} \text{Tr} \left[\gamma^\mu \frac{\partial S_R^{-1}(p)}{\partial p^\mu} \right] \Big|_{p^2 = -\mu^2} = -1 \quad \text{RI/MOM}$$

$$\lim_{m_R \rightarrow 0} \frac{1}{12m_R} \text{Tr}[S_R^{-1}(p)] \Big|_{p^2 = -\mu^2} = 1 \quad \lim_{m_R \rightarrow 0} \frac{1}{12p^2} \text{Tr}[S_R^{-1}(p) \not{p}] \Big|_{p^2 = -\mu^2} = -1 \quad \text{RI'/MOM}$$

Introduction

Ward-Takahashi identities

- Ward-Takahashi identities(WI)

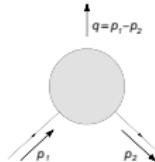
$$q_\mu \Lambda_{V,B}^\mu(p_1, p_2) = S_B^{-1}(p_2) - S_B^{-1}(p_1)$$

$$-iq_\mu \Lambda_{A,B}^\mu(p_1, p_2) = 2m_B \Lambda_{P,B}(p_1, p_2) - i\gamma_5 S_B^{-1}(p_1) - S_B^{-1}(p_2) i\gamma_5$$

- WI valid for renorm. quantities: $O_R = Z_O O$, $\Lambda_{O,R} = \frac{Z_O}{Z_q} \Lambda_{O,B}$

- Renormalization condition on $S \Leftrightarrow$ condition on Λ_O

$$\frac{1}{N} \text{Tr} [\Lambda_{O,R}(p_1, p_2) P_O] \Big|_{\text{mom.conf.}} = 1$$



~~> study quark bilinear operators with **vector**(γ^μ), **axial-vector**($\gamma_5 \gamma^\mu$), **pseudo-scalar**(γ_5) and **scalar**(**1**) operators

- Renormalization constants related:

$$Z_A = 1 = Z_V, Z_P = Z_S, Z_P = 1/Z_m$$

Motivation

■ RI/MOM

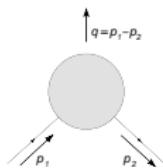
$$\lim_{m_R \rightarrow 0} \frac{1}{48} \text{Tr} \left[\Lambda_{V,R}^\mu(p_1, p_2) \gamma_\mu \right] \Big|_{\text{asym}} = 1, \quad \lim_{m_R \rightarrow 0} \frac{1}{48} \text{Tr} \left[\Lambda_{A,R}^\mu(p_1, p_2) \gamma_5 \gamma_\mu \right] \Big|_{\text{asym}} = 1$$
$$\lim_{m_R \rightarrow 0} \frac{1}{12} \text{Tr} \left[\Lambda_{S,R}(p_1, p_2) \mathbf{1} \right] \Big|_{\text{asym}} = 1, \quad \lim_{m_R \rightarrow 0} \frac{1}{12i} \text{Tr} \left[\Lambda_{P,R}(p_1, p_2) \gamma_5 \right] \Big|_{\text{asym}} = 1$$

■ Asymmetric/exceptional momentum config.(MOM):

$$p_1^2 = p_2^2 = -\mu^2, \quad \mu^2 > 0, \quad p_1 = p_2, \quad q = 0$$

Symmetric/nonexceptional momentum config(SMOM):

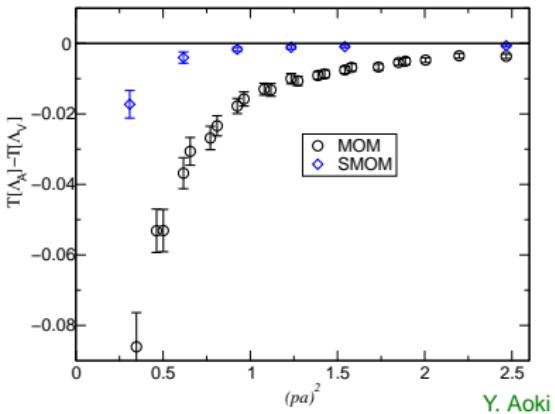
$$p_1^2 = p_2^2 = q^2 = -\mu^2, \quad \mu^2 > 0, \quad q = p_1 - p_2$$



- Need to introduce renormalization scale μ
typically $\mu \sim 2$ GeV for this problem
- Renormalization constants need to be determined through simulation, $m_R = Z_m m_B$

Motivation

- Symmetric subtraction point implies a lattice simulation with suppressed contamination from infrared effects
- For asymmetric subtraction point effects of chiral symmetry breaking vanish slowly like $1/p^2$ for large ext. momenta



- For SMOM infrared effects better behaved, vanishing with larger powers of p N.H. Christ, et al.

Motivation

RI-scheme \implies $\overline{\text{MS}}$ -scheme

- Conversion/Matching factor:

$$m_R^{\overline{\text{MS}}} = C_m^{\text{RI/MOM}} m_R^{\text{RI/MOM}} \quad (C_m \text{ in general gauge dependent})$$

- RI/MOM scheme intermediate scheme before conversion to the $\overline{\text{MS}}$ scheme
- C_m can be computed in cont. PT, e.g. RI/MOM, RI'/MOM: known up to 3-loop G. Martinelli et al.; Franco, Lubicz; Chetyrkin, Retey; Gracey

$$C_m^{\text{RI/MOM}} = 1.0 - 0.1333 - 0.0759 - 0.0557 \quad \alpha_s(2 \text{ GeV})/\pi \sim 0.1$$

$$C_m^{\text{RI'/MOM}} = 1.0 - 0.1333 - 0.0816 - 0.0603 \quad n_f = 3$$

- Observation:

Size of NLO, $N^2\text{LO}$, $N^3\text{LO}$ contr. amount $\sim 13\%$, $\sim 8\%$, $\sim 6\%$

\leadsto poor convergence \leadsto **big error in renormalization**

- Matching to pert. theo.: reduce truncation error: large μ
 \leadsto window problem

Concepts & Framework

RI/SMOM

- Task: Find a RI/MOM type scheme which is "close" to MS scheme, e.g. which has a matching factor with small corrections \leadsto Small expansion coefficients
- Idea: Use subtraction point with symmetric momenta

RI/SMOM conditions:

$$\lim_{m_R \rightarrow 0} \frac{1}{12} \text{Tr} \left[\Lambda_{S,R}(p_1, p_2) \mathbf{1} \right] \Big|_{\text{sym}} = 1, \quad \lim_{m_R \rightarrow 0} \frac{1}{12i} \text{Tr} \left[\Lambda_{P,R}(p_1, p_2) \gamma_5 \right] \Big|_{\text{sym}} = 1$$

$$\lim_{m_R \rightarrow 0} \frac{1}{12q^2} \text{Tr} \left[q_\mu \Lambda_{V,R}^\mu(p_1, p_2) q \right] \Big|_{\text{sym}} = 1, \quad \lim_{m_R \rightarrow 0} \frac{1}{12q^2} \text{Tr} \left[q_\mu \Lambda_{A,R}^\mu(p_1, p_2) \gamma_5 q \right] \Big|_{\text{sym}} = 1$$

- Alternative scheme (RI/SMOM $_{\gamma_\mu}$) using different projectors

$$\lim_{m_R \rightarrow 0} \frac{1}{48} \text{Tr} \left[\Lambda_{V,R}^\mu(p_1, p_2) \gamma_\mu \right] \Big|_{\text{sym}} = 1, \quad \lim_{m_R \rightarrow 0} \frac{1}{48} \text{Tr} \left[\Lambda_{A,R}^\mu(p_1, p_2) \gamma_5 \gamma_\mu \right] \Big|_{\text{sym}} = 1$$

- Ward identity:

Z_q in RI/SMOM the same as RI'/MOM scheme (known to 3-loop)
 \rightarrow need "only" recompute Z_m in RI/SMOM scheme

Results at NLO (order α_s) for RI/SMOM $_{(\gamma_\mu)}$

- Two ways to compute C_m : (follows with WI)

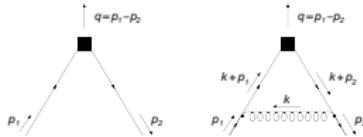
$$1.) (C_m^{\text{RI/SMOM}})^{-1} = (C_m^{\text{RI'/MOM}})^{-1} - \frac{1}{2} C_q^{\text{RI/SMOM}} \lim_{m_R \rightarrow 0} \frac{1}{12 m_R^{\overline{\text{MS}}}} \text{Tr} \left[q_\mu \Lambda_{A,R}^{\mu, \overline{\text{MS}}} \gamma_5 \right] \Big|_{\text{sym}}$$

- 2.) Via pseudo-scalar operator:

$$(C_m^{\text{RI/SMOM}})^{-1} = C_P^{\text{RI/SMOM}} = C_q^{\text{RI/SMOM}} \lim_{m_R \rightarrow 0} \frac{1}{12 i} \text{Tr} \left[\Lambda_{P,R}^{\overline{\text{MS}}} \gamma_5 \right] \Big|_{\text{sym}}$$

- Computation straightforward:

Black box: scalar,
pseudo-scalar, vector,
axial-vector operator



$$\begin{aligned} C_m^{\text{RI/SMOM}} &= 1 - \frac{\alpha_s}{4\pi} C_F [4 + \xi - (3 + \xi) (\frac{1}{3} \Psi'(\frac{1}{3}) - \frac{2}{9} \pi^2)] \\ &= 1 + \frac{\alpha_s}{4\pi} C_F \textcolor{blue}{0.4841391\dots} + \mathcal{O}(\alpha_s^2) \quad (\text{Landau gauge}) \end{aligned}$$

[C_F : color factor, ξ : gauge parameter, Ψ : digamma function]

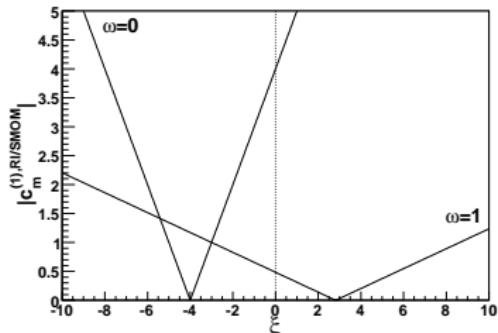
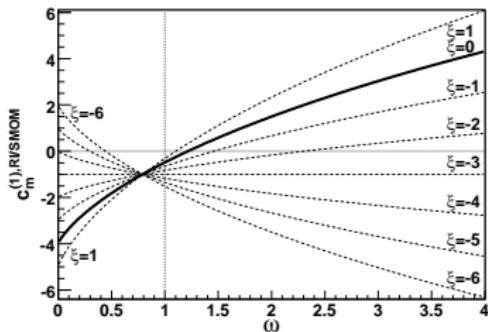
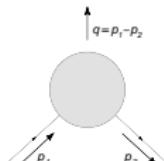
$$C_m^{\text{RI/SMOM}_{\gamma_\mu}} = 1 - \frac{\alpha_s}{4\pi} C_F \textcolor{blue}{1.4841391\dots} + \mathcal{O}(\alpha_s^2) \quad (\text{Landau gauge})$$

- Compared to $C_m^{\text{RI}', \text{RI}} = 1 - \frac{\alpha_s}{4\pi} C_F 4 + \mathcal{O}(\alpha_s^2)$ (Landau gauge)
 $\mathcal{O}(\alpha_s)$ coeff. almost a factor 10(2.7) smaller compared to
RI, RI'; Size of $\mathcal{O}(\alpha_s) \sim 1.6\%(5\%)$ compared to $\mathcal{O}(\alpha_s^3)$ of RI/MOM $\sim 6\%$
- Both methods 1.), 2.) give same result

RI/SMOM result of NLO calculation

- Study different subtraction points and gauge parameter dependence: (subtraction “point” $p_1^2 = p_2^2 = -\mu^2$ and $q^2 = -\omega\mu^2$)

$$C_m = 1 + \frac{\alpha_s}{4\pi} C_F c_m^{(1)}(\omega, \xi)$$

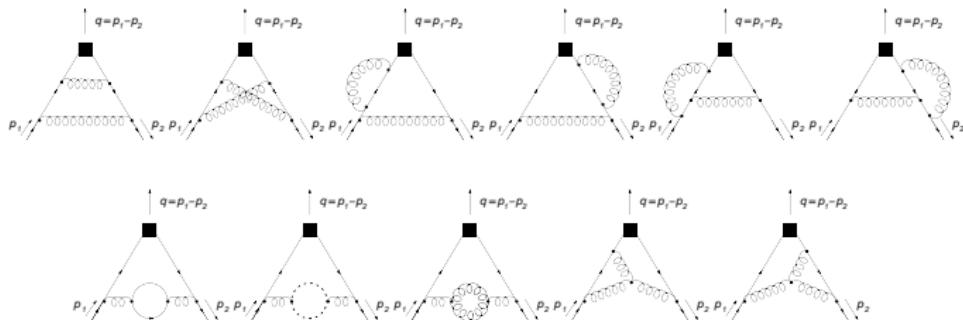


- Symmetric point ($\omega = 1$) almost optimal
- ...if confirmed at higher orders
~~ significant reduction in the calculated value of the quark mass

The NNLO calculation (order α_s^2)

RI/SMOM_($\gamma\mu$) scheme

- Need to compute the matching factor C_m in the new RI/SMOM schemes at two-loop
- At NNLO 11 diagrams:



- Black box:
operator: scalar, pseudo-scalar, vector or axial-vector

Calculation

Techniques

Calculation:

Proceeds in two steps:

■ A) Integration-by-parts (IBP):

K.G. Chetyrkin, F.V. Tkachov

$$0 = \int [d^D \ell_1] \dots [d^D \ell_4] \partial_{(\ell_j)_\mu} (\ell_I^\mu I_{\alpha\beta}) , \quad j, I = 1, \dots, \text{loops}=4$$

$I_{\alpha\beta}$: Generic integrand with propagator powers $\alpha = \{\alpha_1, \dots\}$

and scalar-product powers $\beta = \{\beta_1, \dots\}$

Laporta-Algorithm:

S. Laporta, E. Remiddi

Idea:

- IBP-identities for explicit numerical values of α, β
- Introduction of an order among the integrals
- Solving a linear system of equations

Automation:

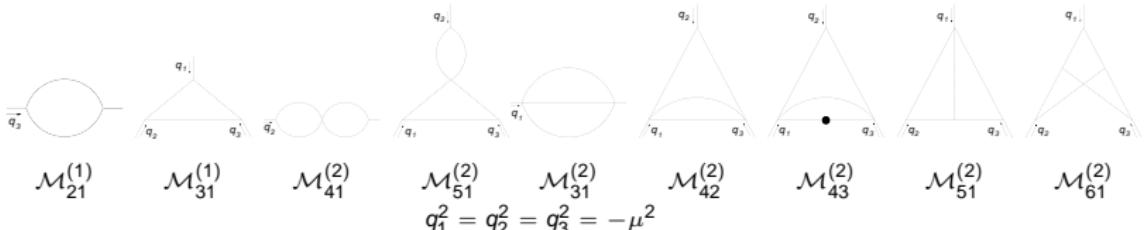
- Generation & solution of the system of lin. equations:
- Implementation based on FORM3 J.A.M. Vermaseren
 - Simplification of rational functions in d by FERMAT

R.H. Lewis

■ B) Solve MI: here known Davydychev et al.

Master Integrals:

- 2 one-loop MI, 7 two-loop MI (two factorized):



- MI known Davydychev et al., results available analytically:

$$\mathcal{M}_{21}^{(1)} = \frac{1}{\varepsilon} + 2 + \varepsilon \left[4 - \frac{\pi^2}{12} \right] + \mathcal{O}(\varepsilon^2),$$

$$\mathcal{M}_{31}^{(1)} = \left(\frac{2}{3} \pi \right)^2 - \frac{2}{3} \Psi' \left(\frac{1}{3} \right) + \varepsilon \left[\frac{12 s_3}{\sqrt{3}} - \frac{35}{108} \frac{\pi^3}{\sqrt{3}} - \frac{\log^2(3) \pi}{4\sqrt{3}} \right] + \varepsilon^2 [\dots] + \mathcal{O}(\varepsilon^3),$$

$$\mathcal{M}_{41}^{(2)} = \frac{1}{\varepsilon^2} + \frac{4}{\varepsilon} + 12 - \frac{\pi^2}{6} + \varepsilon [\dots] + \mathcal{O}(\varepsilon^2),$$

$$\mathcal{M}_{51}^{(2)} = \frac{1}{\varepsilon} \left[\left(\frac{2}{3} \pi \right)^2 - \frac{2}{3} \Psi' \left(\frac{1}{3} \right) \right] + \frac{12 s_3}{\sqrt{3}} - \frac{35}{108} \frac{\pi^3}{\sqrt{3}} - \frac{\log^2(3) \pi}{4\sqrt{3}} - \frac{4}{3} \Psi' \left(\frac{1}{3} \right) + \frac{8}{9} \pi^2 + \varepsilon [\dots] + \mathcal{O}(\varepsilon^3),$$

$$\mathcal{M}_{31}^{(2)} = \frac{1}{4\varepsilon} + \frac{13}{8} + \varepsilon [\dots] + \varepsilon^2 [\dots] + \mathcal{O}(\varepsilon^3), \quad \mathcal{M}_{42}^{(2)} = \dots + \varepsilon [\dots] + \mathcal{O}(\varepsilon^2),$$

$$\mathcal{M}_{43}^{(2)} = \dots + \varepsilon [\dots] + \mathcal{O}(\varepsilon^2), \quad \mathcal{M}_{51}^{(2)} = \dots + \mathcal{O}(\varepsilon), \quad \mathcal{M}_{61}^{(2)} = \dots + \mathcal{O}(\varepsilon).$$

- Higher orders in ε needed due to spurious poles

Results at NNLO and discussion

Results & Comparison of the conversion factors for

$n_f = 3$, $\alpha_s/\pi \simeq 0.1$, scale of ~ 2 GeV

■ RI'/MOM \iff RI/SMOM:

$$C_{m,L}^{\text{RI}'/\text{MOM}} = 1 - 0.1333333\dots - 0.07585848\dots - 0.0556959\dots$$

$$C_{m,L}^{\text{RI}/\text{SMOM}} = 1 - 0.0161380\dots - 0.00660442\dots \leftarrow \text{New}$$

↪ in agreement with Jaeger, Gorbahn

■ RI/MOM \iff RI/SMOM $_{\gamma_\mu}$:

$$C_{m,L}^{\text{RI}/\text{MOM}} = 1 - 0.1333333\dots - 0.0815876\dots - 0.0602759\dots$$

$$C_{m,L}^{\text{RI}/\text{SMOM}_{\gamma_\mu}} = 1 - 0.0494713\dots - 0.0228421\dots \leftarrow \text{New}$$

- Matching factors for schemes with symmetric subtraction point show better convergence behavior

↪ significant reduction of syst. error on light quark masses

- Informative to have multiple schemes

↪ better assessment of syst. errors

- Three-loop anomalous dimension

↪ run quark mass to different energy scales

Summary & Conclusion

- Framework + concepts of renorm. of quark bilinear operators in the RI/SMOM schemes has been discussed
- Results can be used to convert light quark masses in this scheme to the \overline{MS} scheme
- The conversion factors in the RI/SMOM schemes are now available up to NNLO + show a better convergence behavior than in the traditional RI/MOM schemes
- The pert. truncation error is smaller than in RI/MOM
- RI/SMOM schemes are less sensitive to infrared effects in latt. simulation
 - ~~ The use of the RI/SMOM schemes will reduce the systematic error and improve precision of light quark mass determinations from lattice simulations obtained in this approach
- Light quark mass determination in the RI/SMOM schemes still in progress by RBC/UKQCD coll.